JOURNAL OF COMPUTATIONAL PHYSICS **15**, 168–187 (1974)

Analytical Solution in Spherical Geometry of the Monoenergetic S_2 and S_4 Differential Equations, and of the S_n Equations in Vacuum

T. AUERBACH, W. HÄLG, AND J. MENNIG

Institut für Reaktortechnik, Eidg. Technische Hochschule, Zürich/Switzerland

AND

H. BEUCHAT

Eidg. Patentamt, Bern/Switzerland

Received February 26, 1973

The investigation to be presented is based on S_n theory in its original form. Analytical solutions of the S_2 and S_4 differential equations in spherical geometry will be derived, based on solutions in closed form of the S_n equations in vacuum. In the nonvacuum case analytical solutions contain all exponents of r present in the vacuum case, plus an additional term in lnr multiplied by a power series which starts with r^4 in S_2 approximation and with r^{22} in S_4 approximation. The general solution is subject to the symmetry condition at $r = 0$ and will thus contain only terms bounded at the singular point. It will be shown that results go over into the vacuum solution in the limit of vanishing density.

1. INTRODUCTION

In reactor theory the detailed neutron distribution in the core and reflector of a nuclear reactor is governed by the transport equation. Since practical systems are so involved as to preclude rigorous solutions, approximations must be employed in order to derive expressions for the neutron distribution in space and energy. One such approximation is Carlson's S_n theory [1]. The general form of the S_n solutions in cylindrical geometry was first discussed by Beuchat [2]. Mennig [3] derived explicit solutions in spherical geometry.

The S_n method is intrinsically a semianalytical approximation. Subdividing the angular variable into a finite number of intervals changes the neutron transport equation into a system of ordinary differential equations in space. This may be transformed into the fully numerical S_n method on which most codes are based by further discretization of the spatial variables.

Starting from the semianalytical form of S_n theory the present paper considers the problem of finding analytical solutions of the monoenergetic S_n differential equations in spherical geometry. The fully numerical approach avoids difficulties at the singular point $r = 0$ by integrating over a finite volume element. In contrast to this, determination of the analytical solution in the region surrounding $r = 0$ represents the major problem in semianalytical S_n theory. In treating this problem one also has a practical goal in mind: Analytical S_4 calculations with Lie series in plane geometry [4] have led to particularly short computing times. Similar results were obtained from the spherical harmonics program LIE-PN [5] in cylindrical geometry which, like S_n theory, treats singular differential equations in the innermost zone. Hence there exists hope that the analytical approach to spherical geometry will likewise show certain advantages over fully numerical methods. Admittedly, computing times for one-dimensional problems are short, no matter which method is being used. However, the frequent need for performing many successive one-dimensional calculations in a single program provides an incentive for performing each one as rapidly as possible.

Recent versions of Carlson's original S_n formalism have lead to shorter computing times and to improved convergence in purely numerical computations. However, since analytical solutions are independent of this effect, and since the original method is still the one most widely used in text books, it was decided to base the treatment to be presented on the latter. The derivation to be outlined may, of course, be applied to any other set of differential equations corresponding to other modes of angular discretization.

2. Formulation of the Problem

The monoenergetic neutron transport equation for P_1-scattering in spherical geometry [6] is given by

$$\left(\mu \frac{\partial}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} + \Sigma\right) \varphi(r, \mu) = \frac{\Sigma_0}{2} \Phi(r) + \frac{3\Sigma_1}{2} \mu J(r), \qquad (1)$$

where

$$\Sigma_0 = \Sigma_{s0} + \nu\Sigma_f\,, \qquad \Phi(r) = \int_{-1}^{+1} \varphi(r, \mu')\, d\mu',$$

$$\Sigma_1 = \Sigma_{s1}\,, \qquad J(r) = \int_{-1}^{+1} \mu' \varphi(r, \mu')\, d\mu';$$

and where

$$\begin{aligned}
\varphi(r, \mu) &= \text{neutron flux in neutrons/cm}^2\text{ sec},\\
\mu &= \text{cosine of angle between direction of neutron flight and } \vec{r},\\
\Sigma_{s0} &= \text{isotropic part of macroscopic neutron scattering cross-section},\\
\Sigma_{s1} &= \text{linearly anisotropic part of macroscopic neutron scattering}\\
&\quad\ \text{cross-section},\\
\Sigma_f &= \text{macroscopic fission cross-section},\\
\Sigma &= \text{total macroscopic neutron cross-section}.
\end{aligned}$$

The term "P_1-scattering" implies that only the $P_0(\mu)$ and $P_1(\mu)$ terms are kept in an expansion of the scattering cross-section into a series of Legendre polynomials in μ.

If several concentric zones are present φ, Σ, Σ_0, Σ_1, Φ, and J will carry an additional index m:

$$\text{Zone } m: \qquad r_{m-1} \leqslant r \leqslant r_m\,, \qquad m = 1, 2,\ldots, M.$$

For the innermost zone ($m = 1$) $r_0 = 0$. Owing to the singularity at $r = 0$ this zone is of particular interest and finding the neutron distribution in it turns out to be the main problem in solving Eq. (1) for all zones. Lie series solutions may be used in those regions where $r_{m-1} \neq 0$ [2] [5] [7] [8], with series coefficients obtained recursively from the differential equations. Owing to its simplicity the case with $r_{m-1} \neq 0$ will not be discussed further in this paper. The treatment to be presented will thus be confined to the inner zone ($0 \leqslant r \leqslant R$), allowing the zone index to be omitted. In a one-region problem the solution will be valid throughout the entire zone. In contrast to this, a fully numerical treatment requires even single zone problems to be divided into several subzones. The required solution, $\varphi(r, \mu)$, is determined by Eq. (1) and the imposed boundary conditions. The neutron distribution in a spherical reactor of outer radius R must obey the following boundary conditions:

$$\begin{aligned}
\varphi(0, \mu) &= \varphi(0, -\mu),\\
\varphi(R, -\mu) &= 0.
\end{aligned} \tag{2}$$

The first step consists of subdividing the angular variable μ into $n = 2N$ intervals $I_j = [\mu_j\,, \mu_{j+1}]$, and to approximate $\varphi(r, \mu)$ linearly in μ in each interval:

$$\varphi_j(r, \mu) = \frac{\mu_{j+1} - \mu}{\mu_{j+1} - \mu_j}\,\varphi_j(r) + \frac{\mu - \mu_j}{\mu_{j+1} - \mu_j}\,\varphi_{j+1}(r), \tag{3}$$

where

$$j = 1, 2,\ldots .$$

The right-hand side of Eq. (1) is integrated numerically by any of several well-known methods, resulting in the expressions below

$$\Phi(r) = \int_{-1}^{+1} \varphi(r, \mu')\, d\mu' = \sum_{k=1}^{2N+1} w_k^1 \varphi_k(r),$$
$$J(r) = \int_{-1}^{+1} \mu' \varphi(r, \mu')\, d\mu' = \sum_{k=1}^{2N+1} w_k^2 \varphi_k(r), \tag{4}$$

where

$$\varphi_k(r) = \varphi(r, \mu_k).$$

w_k^1 and w_k^2 are weight factors depending on the method of integration employed.

Angular integration over an interval I_j with use of Eq. (3) transforms Eq. (1) into the following set of differential equations [9]:

$$\alpha_{j+1}\varphi'_{j+1}(r) + \beta_{j+1}\varphi_j'(r) + \frac{\gamma_{j+1}}{r}(\varphi_{j+1}(r) - \varphi_j(r)) = \sum_{k=1}^{2N+1} a_{j+1,k}\varphi_k(r) \tag{5}$$

where

$$j = 1, 2,..., 2N,$$

with

$$\alpha_{j+1} = \frac{\mu_j + 2\mu_{j+1}}{6}, \qquad \beta_{j+1} = \frac{\mu_{j+1} + 2\mu_j}{6}, \qquad \gamma_{j+1} = \frac{1 - \mu_{j+1}\mu_j}{\Delta} - \frac{\Delta}{3}$$
$$\Delta = \mu_{j+1} - \mu_j, \qquad \mu_j = -1 + \frac{j-1}{N},$$

where primes denote differentiation with respect to r. In these equations all material constants are contained in the coefficients $a_{j+1,k}$ on the right-hand side:

$$a_{j+1,k} = \frac{\Sigma_0}{2} w_k^1 + \frac{3}{4}\Sigma_1(\mu_{j+1} + \mu_j) w_k^2 - \frac{\Sigma}{2}(\delta_{k,j+1} + \delta_{k,j}),$$
$$j = 1, 2,..., 2N.$$

It will be noted that the set of $2N$ differential equations, Eq. (5), contains $(2N + 1)$ unknowns. For this reason Eq. (5) must be augmented by an additional differential equation obtained by setting $\mu = \mu_1 = -1$ in Eq. (1) [1]. This yields

$$-\varphi_1'(r) + \Sigma\varphi_1(r) = \frac{\Sigma_0}{2}\Phi(r) - \frac{3}{2}\Sigma_1 J(r)$$

or, with use of Eq. (4),

$$-\varphi_1'(r) = \sum_{k=1}^{2N+1} a_{1,k}\varphi_k(r), \qquad a_{1,k} = \frac{\Sigma_0}{2} w_k{}^1 - \frac{3}{2}\Sigma_1 w_k{}^2 - \Sigma\delta_{1,k}\,.$$

The problem now is reduced to solving the following set of S_n differential equations:

$$\left[\varphi_j'(r) + \frac{\kappa_j}{r}\varphi_j(r)\right] + \left[\varphi_{j-1}'(r) - \frac{\kappa_j{}^*}{r}\varphi_{j-1}(r)\right] b_j = \sum_{k=1}^{2N+1} A_{j,k}\varphi_k(r) \tag{6}$$

with

$$b_j = \begin{cases} 0 & j = 1, \\ \dfrac{\beta_j}{\alpha_j} & j > 1, \end{cases} \qquad \alpha_1 = -1, \qquad A_{j,k} = \frac{a_{j,k}}{\alpha_j},$$

$$\kappa_j = \begin{cases} 0 & j = 1, \\ \dfrac{\gamma_j}{\alpha_j} & j > 1, \end{cases} \qquad \kappa_j{}^* = \begin{cases} 0 & j = 1, \\ \dfrac{\kappa_j}{b_j} & j > 1, \end{cases}$$

$$j = 1, 2, \ldots, 2N + 1.$$

After discretization the boundary conditions become

$$\varphi_j(0) = \varphi_{N+1+j}(0), \qquad j = 1, 2, \ldots, N; \tag{7.1}$$

$$\varphi_j(R) = 0, \qquad j = 1, 2, \ldots, N, N + 1. \tag{7.2}$$

3. Vacuum Solution of the S_n Differential Equations

The vacuum is characterized by $\Sigma_0 = \Sigma_1 = \Sigma = 0$. In this special case all $A_{jk} = 0$ and the S_n differential equations become

$$\varphi_1'(r) = 0,$$

$$\varphi_j'(r) + \frac{\kappa_j}{r}\varphi_j(r) + b_j\varphi_{j-1}'(r) - \frac{\kappa_j}{r}\varphi_{j-1}(r) = 0, \qquad j = 2, 3, \ldots\,. \tag{8}$$

The first line of Eq. (8) may be integrated at once, giving

$$\varphi_1(r) = C_1 = \text{const.} \tag{9}$$

Knowing φ_1 it is easy to calculate φ_2, φ_3, etc., since at any step (j) φ_{j-1} is known from the preceding step. The solution of Eq. (8) results in the recursion relation below

$$\varphi_j(r) = C_j r^{-\kappa_j} - b_j \varphi_{j-1}(r) + \kappa_j(1 + b_j)\, r^{-\kappa_j} \int r^{\kappa_j - 1} \varphi_{j-1}(r)\, dr.$$
$$(C_j = \text{integration constant.}) \tag{10}$$

Substituting $\varphi_1 = C_1$ $(j = 2)$ in Eq. (10) one finds for $\varphi_2(r)$

$$\begin{aligned} \varphi_2(r) &= Q_{21} C_1 + Q_{22} C_2 r^{-\kappa_2}, \\ Q_{21} &= Q_{22} = 1. \end{aligned} \tag{11.1}$$

$\varphi_3(r)$ is obtained by substituting $\varphi_2(r)$ into Eq. (10), giving

$$\begin{aligned} &\varphi_3(r) = Q_{31} C_1 + Q_{32} r^{-\kappa_2} C_2 + Q_{33} r^{-\kappa_3} C_3, \\ &Q_{31} = Q_{33} = 1, \qquad Q_{32} = \frac{\kappa_3 + b_3 \kappa_2}{\kappa_3 - \kappa_2} Q_{22}. \end{aligned} \tag{11.2}$$

All 3 solutions $\varphi_1(1)$–$\varphi_3(r)$ are of the form

$$\varphi_j(r) = \sum_{i=1}^{j} Q_{ji} r^{-\kappa_i} C_i, \qquad (\kappa_1 = 0, j = 1, 2, 3). \tag{11.3}$$

It may be shown by a process of complete induction that this form of the solution is valid not only for $j = 1, 2$, and 3, but for every $j \in [1, 2N + 1]$.

Making the replacement

$$k_i = -\kappa_i$$

the solution in vacuum becomes

$$\varphi_j(r) = \sum_{i=1}^{j} Q_{j,i} r^{k_i} C_i, \qquad j \in [1, 2N + 1], \tag{12}$$

with

$$\begin{aligned} Q_{j,1} &= Q_{j,j} = 1, \\ Q_{j,i} &= Q_{j-1,i} \frac{\kappa_j + b_j \kappa_i}{\kappa_j - \kappa_i}, \qquad j \neq i. \end{aligned} \tag{13}$$

Implicit in this solution is the assumption that the denominator of $Q_{j,i}$ differs from zero, i.e.,

$$\prod_{l=i+1}^{j} (\kappa_l - \kappa_i) \neq 0. \tag{14}$$

This condition is always fulfilled by a uniform subdivision of the angular interval ($\Delta\mu_j = \mu_{j+1} - \mu_j = \text{const.}$).

Eq. (12) represents the general vacuum solution of the S_n differential equations, Eq. (8). The solution contains $2N + 1$ integration constants C_i, N of which are eliminated by application of boundary condition (7.1), removing all terms from Eq. (12) for which $k_i < 0$. The remaining $N + 1$ integration constants C_i ($C_{N+j} = 0$ for $j > 1$) allow the $N + 1$ boundary conditions at $r = R$, Eq. (7.2), to be satisfied.

4. Solution of the S_2 Differential Equations with $\Sigma \neq 0$

While the special case of vacuum ("$\Sigma = 0$" stands for $\Sigma_0 = \Sigma_1 = \Sigma = 0$) permits solutions of the S_n differential equations to be written down in closed form, this is no longer possible when $\Sigma \neq 0$ even in the lowest approximation of $n = 2$. The vacuum solution derived in Section 3 does, however, yield valuable information concerning the expected form of the solution when $\Sigma \neq 0$.

Putting $n = 2N = 2$ in Eq. (6) one derives the following set of S_2 equations:

$$\begin{aligned} \varphi_1'(r) &= \sum_{k=1}^{3} A_{1,k}\varphi_k(r) \\ \left[\varphi_2'(r) - \frac{4}{r}\varphi_2(r)\right] + 2\left[\varphi_1'(r) + \frac{2}{r}\varphi_1(r)\right] &= \sum_{k=1}^{3} A_{2,k}\varphi_k(r) \\ \left[\varphi_3'(r) + \frac{2}{r}\varphi_3(r)\right] + \frac{1}{2}\left[\varphi_2'(r) - \frac{4}{r}\varphi_2(r)\right] &= \sum_{k=1}^{3} A_{3,k}\varphi_k(r). \end{aligned} \tag{15}$$

Equation (15) does not appear to have been treated in the literature, and hence nothing is known about the form of its solution. In order to gain some insight into what the solution might be like, we shall first solve the equations in vacuum. Equation (12) for $n = 2$, corresponding to Eq. (15) with $A_{j,k} = 0$, gives

$$\begin{aligned} \varphi_1 &= C_1 \\ \varphi_2 &= C_1 + C_2 r^4 \\ \varphi_3 &= C_1 + C_3\frac{1}{r^2}. \end{aligned} \tag{16}$$

Boundary condition (7.1) requires that

$$\varphi_1(0) = \varphi_3(0)$$

or

$$C_3 = 0. \tag{17}$$

The boundary condition is thus equivalent to requiring $\varphi_j(r)$ to stay bounded at $r = 0$.

Knowledge of the solution in vacuum facilitates the finding of the general solution [i.e. the solution of Eqs. (15) and (7)]. In the limit $\Sigma_0, \Sigma_1, \Sigma \to 0$ the general solution must go over into the vacuum solution:

$$(\text{general solution})_{\Sigma_0,\Sigma_1,\Sigma\to 0} \to \text{vacuum solution.}$$

Such a solution must therefore contain 2 integration constants C_i.

The following ansatz will be made for the general S_2 solution (cf. [2]):

$$\varphi_j(r) = \sum_{\nu=0}^{\infty} f_{j,\nu} r^\nu + (r^4 \ln r) \sum_{\nu=0}^{\infty} h_{j,\nu} r^\nu. \tag{18}$$

Substituting this ansatz into Eq. (6) (with $N = 1$) leads to the following relation:

$$\begin{aligned}&\kappa_j(f_{j,0} - f_{j-1,0})\, r^{-1}\\ &+ \sum_{\nu=1}^{\infty} r^{\nu-1} \left\{[(\nu + \kappa_j) f_{j,\nu} + b_j(\nu - \kappa_j^*) f_{j-1,\nu} + h_{j,\nu-4} + b_j h_{j-1,\nu-4}] - \sum_{k=1}^{3} A_{jk} f_{k,\nu-1}\right\}\\ &+ \ln r \sum_{\nu=1}^{\infty} r^{\nu-1} \left\{[(\nu + \kappa_j)\, h_{j,\nu-4} + b_j(\nu - \kappa_j^*)\, h_{j-1,\nu-4}] - \sum_{k=1}^{3} A_{jk} h_{k,\nu-5}\right\} = 0.\end{aligned} \tag{19.1}$$

The coefficients of r^{-1}, $r^{\nu-1}$ $(\nu > 1)$, and $r^{\nu-1} \ln r$ must vanish separately, resulting in the equations below:

$$\begin{aligned}&f_{j,0} - f_{j-1,0} = 0\\ \nu > 1:\quad &(\nu + \kappa_j) f_{j,\nu} + b_j(\nu - \kappa_j^*) f_{j-1,\nu} + h_{j,\nu-4} + b_j h_{j-1,\nu-4} - \sum_{k=1}^{3} A_{jk} f_{k,\nu-1} = 0\\ \nu \geqslant 1:\quad &(\nu + \kappa_j)\, h_{j,\nu-4} + b_j(\nu - \kappa_j^*)\, h_{j-1,\nu-4} - \sum_{k=1}^{3} A_{jk} h_{k,\nu-5}\,.\end{aligned} \tag{19.2}$$

Since negative indices are not allowed,

$$f_{j,-1} = h_{j,-5} = h_{j,-4} = h_{j,-3} = h_{j,-2} = h_{j,-1} = 0.$$

The first line of Eq. (19.2) implies

$$f_{j,0} = C_1\,,$$

where C_1 is an arbitrary constant. Putting $\nu = 4$ in the second line of Eq. (19.2) and $j = 2$ and 3 shows that the coefficient of $f_{2,4}$ vanishes both times. One is therefore justified in putting

$$f_{2,4} = C_2 ,$$

with C_2 another arbitrary constant.

Since the relations in Eq. (19.2) are linear it follows that all $f_{j,\nu}$ and $h_{j,\nu}$ may be represented by linear combinations of C_1 and C_2. One is therefore led to the following ansatz:

$$\begin{aligned} f_{j,\nu} &= f_{j,\nu}^1 C_1 + f_{j,\nu}^2 C_2 , \\ h_{j,\nu} &= h_{j,\nu}^1 C_1 + h_{j,\nu}^2 C_2 . \end{aligned} \tag{20}$$

From $f_{j,0} = C_1$, $f_{2,4} = C_2$, and the fact that coefficients with negative indices are zero it follows that

$$\begin{aligned} f_{j,0}^1 &= 1, \qquad f_{j,0}^2 = 0, \\ f_{2,4}^1 &= 0, \qquad f_{2,4}^2 = 1, \\ h_{j,-\nu}^1 = h_{j,-\nu}^2 &= f_{j,-\nu}^1 = f_{j,-\nu}^2 = 0 \qquad (\nu > 0). \end{aligned} \tag{20.1}$$

Substitution of Eq. (20) into Eq. (19.2) with the requirement that the resulting equations be satisfied for arbitrary values of C_1 and C_2 leads to 4 equations for the C_j-independent quantities $f_{j,\nu}^1$, $f_{j,\nu}^2$, $h_{j,\nu}^1$, and $h_{j,\nu}^2$:

$$(\nu + \kappa_j) f_{j,\nu}^1 + b_j(\nu - \kappa_j^*) f_{j-1,\nu}^1 + h_{j,\nu-4}^1 + b_j h_{j-1,\nu-4}^1 - \sum_{k=1}^{3} A_{j,k} f_{k,\nu-1}^1 = 0 \tag{21.1}$$

$$(\nu + \kappa_j) f_{j,\nu}^2 + b_j(\nu - \kappa_j^*) f_{j-1,\nu}^2 + h_{j,\nu-4}^2 + b_j h_{j-1,\nu-4}^2 - \sum_{k=1}^{3} A_{j,k} f_{k,\nu-1}^2 = 0 \tag{21.2}$$

$$(\nu + \kappa_j)\, h_{j,\nu-4}^1 + b_j(\nu - \kappa_j^*)\, h_{j-1,\nu-4}^1 - \sum_{k=1}^{3} A_{j,k} h_{k,\nu-5}^1 = 0 \tag{21.3}$$

$$(\nu + \kappa_j)\, h_{j,\nu-4}^2 + b_j(\nu - \kappa_j^*)\, h_{j-1,\nu-4}^2 - \sum_{k=1}^{3} A_{j,k} h_{k,\nu-5}^2 = 0. \tag{21.4}$$

Equation (21) together with the initial conditions gives

$$\begin{aligned} h_{j,\nu}^2 &= 0 \qquad && \text{for all } \nu, \\ f_{j,1}^2 = f_{j,2}^2 = f_{j,3}^2 &= 0 && \text{for all } j. \end{aligned} \tag{22}$$

Equation (20) now simplifies to

$$
\begin{aligned}
f_{j,\nu} &= f^1_{j,\nu} C_1 + f^2_{j,\nu} C_2 , \\
h_{j,\nu} &= h^1_{j,\nu} C_1 .
\end{aligned}
\tag{23}
$$

The functions one is looking for, $\varphi_j(r)$, have the form

$$
\varphi_j(r) = \left(\sum_{\nu=0}^{\infty} f^1_{j,\nu} r^\nu + (r^4 \ln r) \sum_{\nu=0}^{\infty} h^1_{j,\nu} r^\nu \right) C_1 + \left(\sum_{\nu=4}^{\infty} f^2_{j,\nu} r^\nu \right) C_2 \tag{24}
$$

or

$$
\varphi_j(r) = \varphi_j^{1}(r)\, C_1 + \varphi_j^{2}(r)\, C_2 . \tag{25}
$$

In solving for the expansion coefficients $f^1_{j,\nu}$, $f^2_{j,\nu}$, and $h^1_{j,\nu}$, one must distinguish the following 3 cases:

$$
\nu = 1, 2, 3; \qquad \nu = 4; \qquad \nu > 4.
$$

Final results for the S_2 case are

$\nu = 0$: $\quad f^1_{j,0} = 1, \qquad f^2_{j,0} = 0;$

$\nu = 1, 2, 3$: $\quad f^1_{j,\nu}$ from Eq. (I) below; $\qquad f^2_{j,\nu} = 0;$

$\nu = 4$: $\quad f^1_{1,4} = \frac{1}{4} \sum_{k=1}^{3} A_{1,k} f^1_{k,3}\,; \qquad f^1_{2,4} = 0;$

$$
h^1_{2,0} = -12 f^1_{1,4} + \sum_{k=1}^{3} A_{2,k} f^1_{k,3}\,;
$$

$$
f^1_{3,4} = -\frac{h_{2,0}}{12} + \frac{1}{6} \sum_{k=1}^{3} A_{3,k} f^1_{k,3}
$$

$$
f^2_{1,4} = f^2_{3,4} = h^1_{1,0} = h^1_{3,0} = 0; \quad f^2_{2,4} = 1;
$$

$\nu > 4$: $\quad h^1_{j,\nu-4}$ from Eq. II,

$f^1_{j,\nu}$ from Eq. III,

$f^2_{j,\nu}$ from Eq. IV.

$$\text{(I)} \quad f_{j,\nu}^{1} = -\, b_j \left(\frac{\nu - \kappa_j^*}{\nu + \kappa_j}\right) f_{j-1,\nu}^{1} + \frac{1}{\nu + \kappa_j} \sum_{k=1}^{3} A_{j,k} f_{k,\nu-1}^{1} ,$$

$$\text{(II)} \quad h_{j,\nu-4}^{1} = -\, b_j \left(\frac{\nu - \kappa_j^*}{\nu + \kappa_j}\right) h_{j-1,\nu-4}^{1} + \frac{1}{\nu + \kappa_j} \sum_{k=1}^{3} A_{j,k} h_{k,\nu-5}^{1} ,$$

$$\text{(III)} \quad f_{j,\nu}^{1} = -\left[b_j \left(\frac{\nu - \kappa_j^*}{\nu + \kappa_j}\right) f_{j-1,\nu}^{1} + \frac{h_{j,\nu-4}^{1}}{\nu + \kappa_j} + \frac{b_j h_{j-1,\nu-4}^{1}}{\nu + \kappa_j}\right] + \frac{1}{\nu + \kappa_j} \sum_{k=1}^{3} A_{j,k} f_{k,\nu-1}^{1} ,$$

$$\text{(IV)} \quad f_{j,\nu}^{2} = -\left[b_j \left(\frac{\nu - \kappa_j^*}{\nu + \kappa_j}\right) f_{j-1,\nu}^{2} + \frac{h_{j,\nu-4}^{2}}{\nu + \kappa_j} + \frac{b_j h_{j-1,\nu-4}^{2}}{\nu + \kappa_j}\right] + \frac{1}{\nu + \kappa_j} \sum_{k=1}^{3} A_{j,k} f_{k,\nu-1}^{2} \qquad (j = 1, 2, 3).$$

This completes the derivation. All coefficients $f_{j,\nu}$ and $h_{j,\nu}$ are seen to be determined unambiguously. Moreover, the solution is bounded at the origin, thus satisfying boundary condition (7.1), and contains 2 arbitrary constants C_1 and C_2 needed for satisfying boundary condition (7.2). It is, therefore, the complete solution of Eq. (15).

It may be argued that a more general ansatz of the form

$$\varphi_j(r) = \sum_{\nu=0}^{\infty} f_{j,\nu} r^{\nu} + (r^4 \ln^p r) \sum_{\nu=0}^{\infty} h_{j,\nu} r^{\nu}$$

with p some integer could equally well have been used in place of Eq. (18). However, for $p \neq 1$ all $h_{j,\nu}$ turn out to be zero, so that the resulting $\varphi_j(r)$ is not the most general solution. The latter requires p to equal 1, corresponding to Eq. (18).

5. Solution of the S_4 Differential Equations for $\Sigma \neq 0$

The S_4 approximation is obtained by putting $n = 2N = 4$. According to Eq. (6) the S_4 differential equations are

$$\varphi_1'(r) = \sum_{k=1}^{5} A_{1,k} \varphi_k(r)$$

$$\left[\varphi_2'(r) - \frac{5}{2} \frac{1}{r} \varphi_2(r)\right] + \frac{5}{4} \left[\varphi_1'(r) + \frac{2}{r} \varphi_1(r)\right] = \sum_{k=1}^{5} A_{2,k} \varphi_k(r)$$

$$\left[\varphi_3'(r) - \frac{22}{r}\varphi_3(r)\right] + 2\left[\varphi_2'(r) + \frac{11}{r}\varphi_2(r)\right] = \sum_{k=1}^{5} A_{3,k}\varphi_k(r)$$

$$\left[\varphi_4'(r) + \frac{11}{r}\varphi_4(r)\right] + \frac{1}{2}\left[\varphi_3'(r) - \frac{22}{r}\varphi_3(r)\right] = \sum_{k=1}^{5} A_{4,k}\varphi_k(r)$$

$$\left[\varphi_5'(r) + \frac{2}{3}\varphi_5(r)\right] + \frac{4}{5}\left[\varphi_4'(r) - \frac{5}{2r}\varphi_4(r)\right] = \sum_{k=1}^{5} A_{5,k}\varphi_k(r). \tag{26}$$

A solution of these equations is again suggested by the vacuum solution in analogy with the approach employed in S_2 theory. The S_n solution in vacuum, Eq. (12), for $n = 4$ becomes

$$\begin{aligned}
\varphi_1(r) &= C_1\,, \\
\varphi_2(r) &= C_1 + r^{5/2}C_2\,, \\
\varphi_3(r) &= C_1 + \tfrac{18}{13}r^{5/2}C_2 + r^{22}C_3\,, \\
\varphi_4(r) &= C_1 + r^{5/2}C_2 \qquad\qquad + r^{-11}C_4\,, \\
\varphi_5(r) &= C_1 \qquad\qquad\qquad\quad - \tfrac{6}{5}r^{-11}C_4 + r^{-2}C_5\,.
\end{aligned} \tag{27}$$

Boundary conditions (7.1) at $r = 0$ require that

$$\begin{aligned}
\varphi_1(0) &= \varphi_5(0), \\
\varphi_2(0) &= \varphi_4(0).
\end{aligned} \tag{28}$$

Applying this to the vacuum solution of Eq. (27) implies

$$C_4 = C_5 = 0$$

so that the vacuum solution that is bounded at the origin becomes

$$\varphi_j^{\text{vac}}(r) = C_1 + r^{5/2}Q_{j,2}C_2 + r^{22}Q_{j,3}C_3\,.$$

One now makes the following ansatz for the solution of Eq. (26), satisfying boundary condition (28):

$$\varphi_j(r) = \sum_{\nu=0}^{\infty} f_{j,\nu}r^{\nu} + r^{5/2}\sum_{\nu=0}^{\infty} g_{j,\nu}r^{\nu} + (r^{22}\ln r)\sum_{\nu=0}^{\infty} h_{j,\nu}r^{\nu}. \tag{29}$$

The S_4 differential equations yield relations from which the expansion coefficients $f_{j,\nu}$, $g_{j,\nu}$, and $h_{j,\nu}$ may be determined:

$$(\nu + \kappa_j)f_{j,\nu} + b_j(\nu - \kappa_j^*)f_{j-1,\nu} + h_{j,\nu-22} + b_j h_{j-1,\nu-22} - \sum_{k=1}^{5} A_{j,k}f_{k,\nu-1} = 0, \quad (30.1)$$

$$(\nu + \kappa_j + \tfrac{5}{2})\, g_{j,\nu} + b_j(\nu - \kappa_j^* + \tfrac{5}{2})\, g_{j-1,\nu} - \sum_{k=1}^{5} A_{j,k}g_{k,\nu-1} = 0, \quad (30.2)$$

$$(\nu + \kappa_j)\, h_{j,\nu-22} + b_j(\nu - \kappa_j^*)\, h_{j-1,\nu-22} - \sum_{k=1}^{5} A_{j,k}h_{k,\nu-23} = 0, \quad (30.3)$$

$$(j = 1, 2, 3, 4, 5; f_{j,-\nu} = g_{j,-\nu} = h_{j,-\nu} = 0).$$

Initial values $f_{j,0}$, $g_{j,0}$, and $h_{j,0}$, needed for solving the recursion relations (30), are obtained from Eqs. (30.1) and (30.2) by putting $\nu = 0$, and from Eqs. (30.1) and (30.3) by putting $\nu = 22$.

$\nu = 0$, Eq. (30.1)

$b_1 = 0$, $j = 1 \rightarrow 0 \cdot f_{1,0} = 0 \rightarrow f_{1,0} = C_1$, (1st integration constant)

$j = 2, 3, 4, 5 \rightarrow f_{2,0} = f_{3,0} = f_{4,0} = f_{5,0} = C_1$; (31)

$\nu = 0$, Eq. (30.2)

$j = 1 \rightarrow g_{1,0} = 0$,

$j = 2 \rightarrow 0 \cdot g_{2,0} = 0 \rightarrow g_{2,0} = C_2$, (2nd integration constant)

$j = 3, 4, 5 \rightarrow g_{3,0} = \frac{18}{13}C_2$, $g_{4,0} = C_2$, $g_{5,0} = 0$. (32)

On the one hand, initial values of $g_{j,0}$ depend only on C_2, on the other, recursion relations for $g_{j,\nu}$ involve neither $f_{j,\nu}$ nor $h_{j,\nu}$. It is clear, therefore, that $g_{j,\nu}$ can depend only on C_2 and that one can put

$$g_{j,\nu} = g_{j,\nu}^2 C_2 . \quad (33)$$

Substituting this into Eq. (30.1) and noting that $(\nu + \kappa_j + \frac{5}{2}) \neq 0$ one finds the recursion relation for $g_{j,\nu}^2$ below, valid for $\nu > 0$

$$g_{j,\nu}^2 = -b_j \left(\frac{\nu - \kappa_j^* + \frac{5}{2}}{\nu + \kappa_j + \frac{5}{2}}\right) g_{j-1,\nu}^2 + \frac{1}{\nu + \kappa_j + \frac{5}{2}} \sum_{k=1}^{5} A_{j,k} g_{k,\nu-1}^2 . \quad (34)$$

Since, $h_{j,-\nu} = 0$ for $1 \leqslant \nu \leqslant 21$ Eq. (30.1) simplifies in this range to

$$(\nu + \kappa_j) f_{j,\nu} + b_j(\nu - \kappa_j^*) f_{j-1,\nu} - \sum_{k=1}^{5} A_{j,k} f_{k,\nu-1} = 0. \tag{35}$$

In this interval of ν the coefficient $f_{j,\nu}$ depends only on $f_{5,0} = C_1$. Hence for $1 \leqslant \nu \leqslant 21$ one puts

$$f_{j,\nu} = f^1_{j,\nu} C_1 . \tag{36}$$

Eqs. (35) and (36) together give

$$f^1_{j,\nu} = -b_j \left(\frac{\nu - \kappa_j^*}{\nu + \kappa_j}\right) f^1_{j-1,\nu} + \frac{1}{\nu + \kappa_j} \sum_{k=1}^{5} A_{j,k} f^1_{k,\nu-1} . \tag{37}$$

The case of $\nu = 22$ must be discussed separately, because in this instance $(\nu + \kappa_j) = 0$ for $j = 3$:

$\nu = 22,$ Eq. (30.3)

$$j = 1, 2 \to h_{1,0} = 0, \qquad h_{2,0} = 0;$$
$$j = 3 \quad \to 0 \cdot h_{3,0} = 0.$$

This provides no information about $h_{3,0}$. Putting

$$h_{3,0} = C_3^*,$$
$$j = 4, 5 \to h_{4,0} = h_{5,0} = 0.$$

$h_{3,0} = C_3^*$ is obtained from Eq. (30.1):

$$f_{1,22} = \left(\frac{1}{22} \sum_{k=1}^{5} A_{1,k} f^1_{k,21}\right) C_1 \equiv f^1_{1,22} C_1 ,$$

$$f_{2,22} = \left(-\frac{20}{13} f^1_{1,22} + \frac{2}{39} \sum_{k=1}^{5} A_{2,k} f^1_{k,21}\right) C_1 \equiv f^1_{2,22} C_1 ,$$

$$0 \cdot f_{3,22} + 66 f^1_{2,22} C_1 + h_{3,0} - \left(\sum_{k=1}^{5} A_{3,k} f^1_{k,21}\right) C_1 = 0.$$

$f_{3,22}$ cannot be determined from this equation because its coefficient is zero. One therefore puts

$$f_{3,22} = C_3 \qquad \text{(3rd integration constant)}.$$

Furthermore, one has

$$h_{3,0} \equiv C_3{}^* = \left[-66f^1_{2,22} + \sum_{k=1}^{5} A_{3,k} f^1_{k,21}\right] C_1 \equiv h^1_{3,0} C_1 ,$$

$$f_{4,22} = \left(-\frac{1}{66} h^1_{3,0} + \frac{1}{33} \sum_{k=1}^{5} A_{4,k} f^1_{k,21}\right) C_1 \equiv f^1_{4,22} C_1 ,$$

$$f_{5,22} = \left(-\frac{13}{20} f^1_{4,22} + \frac{1}{24} \sum_{k=1}^{5} A_{5,k} f^1_{k,21}\right) C_1 \equiv f^1_{5,22} C_1 .$$

All quantities $f_{j,22}$ are of the form

$$f_{j,22} = f^1_{j,22} C_1 + f^3_{j,22} C_3 , \tag{38}$$

with

$$f^1_{1,22} = \frac{1}{22} \sum_{k=1}^{5} A_{1,k} f^1_{k,21} , \qquad f^3_{1,22} = 0;$$

$$f^1_{2,22} = -\frac{20}{13} f^1_{1,22} + \frac{2}{39} \sum_{k=1}^{5} A_{2,k} f^1_{k,21} , \qquad f^3_{2,22} = 0;$$

$$f^1_{3,22} = 0, \qquad h^1_{3,0} = -66 f^1_{22} + \sum_{k=1}^{5} A_{3,k} f^1_{k,21} , \quad f^3_{3,22} = 1;$$

$$f^1_{4,22} = -\frac{1}{66} h^1_{3,0} + \frac{1}{33} \sum_{k=1}^{5} A_{4,k} f^1_{k,21} , \qquad f^3_{4,22} = 0;$$

$$f^1_{5,22} = -\frac{13}{20} f^1_{4,22} + \frac{1}{24} \sum_{k=1}^{5} A_{5,k} f^1_{k,21} , \qquad f^3_{5,22} = 0;$$

and all $h_{j,0}$ have the form

$$h_{j,0} = h^1_{j,0} C_1 , \tag{39}$$

with

$$h^1_{j,0} = \begin{cases} 0 & j \neq 3; \\ -66 f^1_{2,22} + \sum_{k=1}^{5} A_{3,k} f^1_{k,21} , & j = 3. \end{cases}$$

$\nu > 22$:

In this case $(\nu + \kappa_j) \neq 0$ and the previously determined values of $h_{j,0}$ now permit recursive calculation of all $h_{j,\nu}$. $h_{j,0}$ is proportional to C_1, according to Eq. (39), and $h_{j,\nu}$ is obtained recursively from $h_{j,0}$. Hence all $h_{j,\nu}$ are also proportional to C_1 and one can make the ansatz

$$h_{j,\nu-22} = h^1_{j,\nu-22} C_1 . \tag{40}$$

Eqs. (30.3) and (40) then give

$$h^1_{j,\nu-22} = -b_j \left(\frac{\nu - \kappa_j{}^*}{\nu + \kappa_j}\right) h^1_{j-1,\nu-22} + \frac{1}{\nu + \kappa_j} \sum_{k=1}^{5} A_{j,k} h^1_{k,\nu-23} .$$

The constants $f_{j,\nu}$ for $\nu > 22$ are found in similar manner by an extension of the form found for $\nu = 22$:

$$f_{j,\nu} = f^1_{j,\nu} C_1 + f^3_{j,\nu} C_3 . \tag{41}$$

Substitution of $f_{j,\nu}$, Eq. (41), into Eq. (30.1) and solving for $f^1_{j,\nu}$ and $f^3_{j,\nu}$ results in the relation

$$\begin{aligned} f^1_{j,\nu} &= -b_j \left(\frac{\nu - \kappa_j{}^*}{\nu + \kappa_j}\right) f^1_{j-1,\nu} - \frac{h^1_{j,\nu-22}}{\nu + \kappa_j} - b_j \frac{h^1_{j-1,\nu-22}}{\nu + \kappa_j} + \frac{\sum_{k=1}^{5} A_{j,k} f^1_{k,\nu-1}}{\nu + \kappa_j} , \\ f^3_{j,\nu} &= -b_j \left(\frac{\nu - \kappa_j{}^*}{\nu + \kappa_j}\right) f^3_{j-1,\nu} + \left(\sum_{k=1}^{5} A_{j,k} f^3_{k,\nu-1}\right) \frac{1}{\nu + \kappa_j} . \end{aligned} \tag{42}$$

With initial values $f^1_{j,22}$ and $f^3_{j,22}$ known from previous calculations, and with $(\nu + \kappa_j) \neq 0$ for all $\nu > 22$, all $f^1_{j,\nu}$ and $f^3_{j,\nu}$ may be determined recursively-from Eq. (42).

The solution $\varphi_j(r)$ now presents itself in the following form:

$$\varphi_j(r) = \left(\sum_{\nu=0}^{\infty} (f_{j,\nu} + r^{22}(\ln r) h_{j,\nu}) r^\nu\right) C_1 + r^{5/2} \left(\sum_{\nu=0}^{\infty} g^2_{j,\nu} r^\nu\right) C_2 + \left(\sum_{\nu=22}^{\infty} f^3_{j,\nu} r^\nu\right) C_3$$

or

$$\varphi_j(r) = \varphi_j{}^1(r)\, C_1 + \varphi_j{}^2(r)\, C_2 + \varphi_j{}^3(r)\, C_3 .$$

An interesting feature of this result is the fact that neither $\varphi_j{}^1(r)$ nor $\varphi_j{}^2(r)$ are expressible in terms of pure power series alone. $\varphi_j{}^2(r)$ is nondifferentiable at $r = 0$

and $\varphi_j^1(r)$ possesses only derivatives of order 1 to 21 at this point. However, both $\varphi_j^1(r)$ and $\varphi_j^2(r)$ are bounded at $r = 0$:

$$\lim_{r\to 0} \varphi_j^1(r) = f_{j,0}(0) = 1,$$
$$\lim_{r\to 0} \varphi_j^2(r) = 0.$$

In the special case of $\Sigma_0 = \Sigma_1 = \Sigma = 0$ ($A_{jk} = 0$), i.e. for vacuum, one finds

$$h_{j,\nu} = 0,$$
$$f^1_{j,\nu} = 0 \qquad (\nu > 0),$$
$$f^3_{j,22} = \begin{cases} 0 & j \neq 22, \\ 1 & j = 22, \end{cases}$$
$$g_{j,\nu} = 0 \qquad (\nu > 0).$$

For $\Sigma_0 = \Sigma_1 = \Sigma = 0$ the general solution is thus seen to go over into the vacuum solution, as it should:

$$\begin{aligned}
\varphi_1(r) &= C_1\,, \\
\varphi_2(r) &= C_1 + r^{5/2}C_2\,, \\
\varphi_3(r) &= C_1 + \tfrac{18}{13}r^{5/2}C_2 + r^{22}C_3\,, \\
\varphi_4(r) &= C_1 + r^{5/2}C_2\,, \\
\varphi_5(r) &= C_1\,.
\end{aligned}$$

6. Generalization

Comparison of the vacuum solution ($\Sigma = 0$) with the general solution ($\Sigma \neq 0$) in S_2 and S_4 approximation suggests a way to proceed in the general case of $\Sigma \neq 0$ and n arbitrary, as indicated in Table I.

TABLE I

n	$\Sigma = 0$ (vacuum)	$\Sigma \neq 0$
2	$\varphi_1 = \varphi_3 = C_1$ $\varphi_2 = C_1 + r^4 C_2$	$\varphi_j(r) = f_j(r) + (r^4 \ln r)\, h_j(r)$
4	$\varphi_1 = \varphi_5 = C_1$ $\varphi_2 = \varphi_4 = C_1 + r^{5/2}C_2$ $\varphi_3 = C_1 + \frac{18}{13} r^{5/2} C_2 + r^{22} C_3$	$\varphi = f_j(r) + r^{5/2} g_j(r) + (r^{22} \ln r)\, h_j(r)$

The vacuum solution of the S_n differential equations (Eq. 12) satisfying the boundary condition at $r = 0$ is

$$\varphi_j^{\text{vac}}(r) = \sum_{i=1}^{j} Q_{j,i} C_i r^{k_i}, \tag{43}$$

where only $k_i \geqslant 0$ enter into the series and $C_{N+l} = 0$ for $l > 1$. Starting from this solution, the general solution in S_2 and S_4 approximation was obtained in two steps:

1. $Q_{j,i}C_i$ is replaced by a power series $F_{j,i}(r)$ in all terms of φ_j^{vac} whose k_i are not integers.
2. $Q_{j,i}C_i$ is replaced by a product of a power series and $(\ln r)^{M_r}$ in terms where k_i is an integer.

More than a single integer k_i can occur in approximations with $n > 4$, as shown in Table II below. ($k_i = 0$, the heavy line separates positive values of k_i from negative ones).

Whenever more than one integer k_i occurs a factor $(\ln r)^{M_r}$ should be introduced, where M_r is equal to the order number of k_i arranged in a sequence of increasing magnitude (index r). Apart from this, the two generalizing steps derived on the basis of S_2 and S_4 solutions are retained. Applying this to the higher order S_n solutions and using the table of k_i values one finds:

$$S_6:\ \varphi_j(r) = F_{j,1}(r) + r^{2\frac{2}{7}} F_{j,2}(r) + (r^{10} \ln r) F_{j,3}(r) + [r^{52} (\ln r)^2] F_{j,4}(r),$$

$$S_8:\ \varphi_j(r) = F_{j,1}(r) + r^{2\frac{1}{5}} F_{j,2}(r) + r^{8\frac{2}{7}} F_{j,3}(r) + r^{20\frac{1}{2}} F_{j,4}(r) + (r^{94} \ln r) F_{j,5}(r),$$

$$S_{10}:\ \varphi_j(r) = F_{j,1}(1) + r^{2\frac{2}{13}} F_{j,2}(r) + r^{7\frac{3}{5}} F_{j,3}(r) + (r^{16} \ln r) F_{j,4}(r) + [r^{34} (\ln r)^2] F_{j,5}(r) + [r^{148} (\ln r)^3] F_{j,6}(r),$$

$$S_{12}:\ \varphi_j(r) = F_{j,1}(r) + r^{2\frac{1}{8}} F_{j,2}(r) + r^{7\frac{3}{13}} F_{j,3}(r) + r^{14\frac{1}{5}} F_{j,4}(r) + r^{25\frac{3}{7}} F_{j,5}(r) + r^{50\frac{1}{2}} F_{j,6}(r) + (r^{214} \ln r) F_{j,7}(r), \qquad \text{etc.}$$

with

$$F_{j,i}(r) = \sum_{\nu=0}^{\infty} F_{j,i,\nu} r^{\nu}.$$

In the general case (n = arbitrary) the functions $F_{j,i}(r)$ will contain a total of $N + 1$ integration constants C_i so that an alternative way of expressing $\varphi_j(r)$ is

$$\varphi_j^{\text{vac}}(r) = \sum_{i=1}^{N+1} \varphi_{j,i}(r)\, C_i\,; \qquad j = 1, 2, \ldots, 2N + 1. \tag{44}$$

TABLE II

$S_n \rightarrow$	S_2	S_4	S_6	S_8	S_{10}	S_{12}	S_{14}	S_{16}	S_{18}	S_{20}
$n/2=N\rightarrow$	1	2	3	4	5	6	7	8	9	10
$j\downarrow$	k_j	k_j	k_j	k_j	k_j	k_j	k_j	k_j	k_j	k_j
2	4	$2\frac{1}{2}$	$2\frac{2}{7}$	$2\frac{1}{5}$	$2\frac{2}{13}$	$2\frac{1}{8}$	$2\frac{2}{19}$	$2\frac{1}{11}$	$2\frac{2}{25}$	$2\frac{1}{14}$
3	-2	22	10	$8\frac{2}{7}$	$7\frac{3}{5}$	$7\frac{3}{13}$	7	$6\frac{16}{19}$	$6\frac{8}{11}$	$6\frac{16}{25}$
4		-11	52	$20\frac{1}{2}$	16	$14\frac{1}{5}$	$13\frac{3}{13}$	$12\frac{5}{8}$	$12\frac{4}{9}$	$11\frac{10}{11}$
5		-2	-26	94	34	$25\frac{3}{7}$	22	$20\frac{2}{13}$	19	$18\frac{4}{19}$
6			-8	-47	148	$50\frac{1}{2}$	$36\frac{4}{7}$	31	28	$26\frac{1}{8}$
7			-2	$-16\frac{2}{5}$	-74	214	70	$49\frac{3}{7}$	$41\frac{1}{5}$	$34\frac{10}{13}$
8				$-7\frac{1}{4}$	$-27\frac{1}{5}$	-107	292	$92\frac{1}{2}$	64	$52\frac{3}{5}$
9				-2	-14	$-40\frac{2}{5}$	-146	382	118	$80\frac{2}{7}$
10					$-6\frac{10}{11}$	$-22\frac{1}{4}$	-56	-191	484	$146\frac{1}{2}$
11					-2	$-12\frac{10}{11}$	-32	-74	-242	598
12						$-6\frac{5}{7}$	-20	$-43\frac{1}{4}$	$-92\frac{2}{5}$	-299
13						-2	$-12\frac{2}{7}$	$-28\frac{2}{11}$	-56	$-117\frac{1}{5}$
14							$-6\frac{10}{17}$	$-18\frac{5}{7}$	$-37\frac{5}{11}$	$-70\frac{1}{4}$
15							-2	$-11\frac{15}{17}$	-26	$-47\frac{9}{11}$
16								$-6\frac{1}{2}$	$-17\frac{15}{17}$	$-34\frac{1}{7}$
17								-2	$-11\frac{3}{5}$	$-24\frac{10}{17}$
18									$-6\frac{10}{23}$	$-17\frac{3}{10}$
19									-2	$-11\frac{9}{23}$
20										$-6\frac{5}{13}$
21										-2

The discussion in this Section has been confined to a description of the general procedure because an explicit derivation of the general solution for arbitrary n would involve a prohibitively complicated mathematical formalism, as may be judged by the S_4 solution.

CONCLUDING REMARKS

It has been shown that analytical solutions may be derived for the space dependence of the angular neutron distribution in S_n theory. Since the mathematical effort involved increases rapidly with increasing n, practical considerations limit the order of S_n theory that can reasonably be treated analytically to relatively low values of n. Practice has shown, however, that a low order theory like S_4 suffices for calculating the neutron flux in most nuclear reactors.

When n is large, one might preferably employ S_∞ theory [10] [11], which, in a sense, is complementary to S_n theory. While the latter subdivides μ into discrete intervals and leaves r analytic, the former subdivides the spatial variable into discrete intervals and leaves μ analytic. S_∞ theory has so far been restricted to plane geometry. Extending it to spherical geometry should be of considerable interest.

REFERENCES

1. B. G. CARLSON, Los Alamos Scientific Laboratory, Reports LA-1599 (1953) and LA-1891 (1955).
2. H. BEUCHAT, Zur analytischen Lösung der energieabhängigen Neutronen-Transportgleichung in S_N-Formulierung, Thesis, University of Berne, (1966).
3. J. MENNIG (unpublished).
4. T. AUERBACH, J.-P. GANDILLON, W. HÄLG, AND J. MENNIG, Analytical solution of S_4-equations in plane geometry, *Computer Methods in Applied Mechanics and Engineering* **2** (1973), 133–146.
5. R. ALPIAR, LIE-PN, a P_n-spherical harmonics, multi-group, multi-region programme, EIR-110, Eidg. Inst. f. Reaktorforschung, Würenlingen (1967).
6. A. WEINBERG AND E. WIGNER, "The Physical Theory of Neutron Chain Reactors," The University of Chicago Press, (1958).
7. J. MENNIG AND T. AUERBACH, The application of Lie series to reactor theory, *Nucl. Sci. Eng.* **28** (1967), 159–165.
8. J. MENNIG, The solution of ordinary differential equations by means of Lie series; EIR-114, Eidg. Inst. f. Reaktorforschung, Würenlingen (1967).
9. B. G. CARLSON AND G. I. BELL, "Solution of the Transport Equation by the S_n Method," United Nations Geneva Conf. Vol. 16/III (1958).
10. J. MENNIG AND J. T. MARTI, S_∞ neutron transport theory in plane geometry, *Nucl. Sci. Eng.* **31** (1968), 365–368.
11. P. ERDÖS, S. B. HALEY, J. T. MARTI, AND J. MENNIG, A new method for the solution of the transport equation in slab geometry, *J. Comput. Phys.* **6** (1970), 29.